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Hierarchical spatial organization of geographical networks

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Abstract
In this work, we propose a hierarchical extension of the polygonality index as the means to characterize geographical planar networks. By considering successive neighborhoods around each node, it is possible to obtain more complete information about the spatial order of the network at progressive spatial scales. The potential of the methodology is illustrated with respect to synthetic and real geographical networks.

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1. Introduction

Geographical complex networks have been used to model a number of real world systems, including airports, railways, electric power grids, urban roads, urban streets, among others [1]. In particular, these networks have also been used to model several biological systems, such as bone structures [2] and interconnectivity between mammalian cortical areas [3]. In these systems, the physical proximity and communication between neighboring elements are critical in order to guarantee proper development and biological function. It is also known (e.g. [4]) that many biological systems depend on the proper adjacency of cells for correct development, e.g. cell communication, distribution of cells in retina, kidney structures, among many others.

Despite the large set of tools and measurements used to characterize the networks that underlie these systems [5], relatively little attention has been given to the spatial organization of the nodes. In order to address this issue, we propose the use of the polygonality index [6, 7]—a robust measurement able to quantify the spatial order of systems of points—over the nodes of a geographical planar network. In its original version [6], the polygonality index provides a local quantification (i.e. with respect to each node) of the uniformity of the distribution of the angles between the reference node and its immediate neighbors. In this work we extend that concept in order to account for progressive concentric neighborhoods (the hierarchical neighborhoods) around each node.

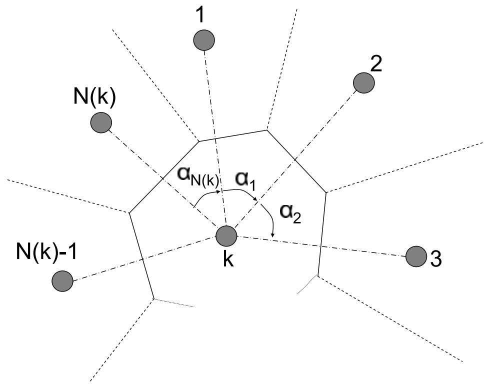


Figure 1. Quantifying the spatial organization around a node. All the angles α_i formed between successive neighbors $(1, \ldots, N(k))$ of the node k are used to compute the polygonality index (see equations (1) and (2)).

This paper starts by presenting, in section 2, a short description of the polygonality index and its extension to subsequent hierarchical levels of the network. Section 3 presents some applications of the proposed methodology, considering a synthetic and a real geographical networks.

2. Hierarchical polygonality index

The polygonality index [6] assigns to each node of the network a value that indicates the amount of the angular organization around the node, with respect to the connections with its immediate neighbors. This value is computed based on the angle α_i formed between successive neighbors of a node, as exemplified in figure 1 and expressed by the following equation:

$$\Delta(k) = \frac{1}{\sum_{i=1}^{N(k)} |\alpha_i - \beta_k| + 1}, \tag{1}$$

where k is the node under analysis, $N(k)$ is the number of immediate neighbors of the node k and β_k is a parameter whose value can be fixed so as to characterize specific spatial arrangements (e.g. $\beta_k = \pi/3$ for the characterization of hexagonality), or can vary accordingly to the number of neighbors of the nodes, i.e., $\beta_k = 2\pi/N(k)$. A fixed value of β_k allows one to identify whether the spatial position of the adjacent nodes obeys a specific arrangement (hexagonal, orthogonal, etc). On the other hand, when the value of β_k depends on the number of neighbors, the polygonality index indicates how well the angles between the neighbors are equally distributed. In both cases, the polygonality index varies between 0 (total lack of angular order) and 1 (fully organized system). Because of the intrinsic relationship between angular and spatial organization of the points (or nodes), a high level of angular order tends to imply spatial order. In other words, it is virtually impossible to obtain a spatially irregular distribution of points which has high angular order. It should also be observed that the polygonality index was originally proposed to characterize systems of isolated points, whose connectivity is needed to be inferred by establishing a neighborhood system (e.g. by using the Voronoi tessellation). In the case of geographical networks, the neighborhoods are intrinsically provided by the respective topology of the network.

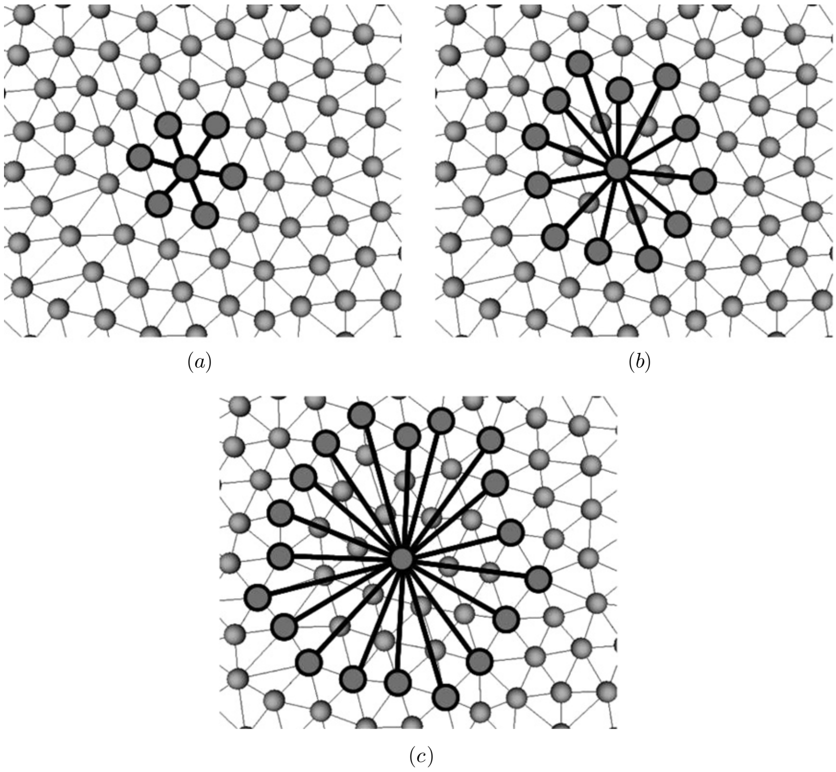


Figure 2. Hierarchical neighbors used to estimate the spatial organization. Hierarchical neighbors 1 (*a*), 2 (*b*) and 3 (*c*). The virtual edges (thicker lines) between the central node and its hierarchical neighbors are used to estimate the angles α_i for the polygonality index computation (see equation (2)).

In this paper, we extend the concept of the polygonality index in order to allow the quantification of the angular and spatial order in a complex network while considering several hierarchical levels around each node. Note that the connections between a node p and its successive neighborhoods define a tree (therefore a hierarchy of levels), so that the first neighbors belong to the first hierarchy (first level of the tree), the second neighbors belong to the second hierarchy (second level of the tree) and so on. For the extension of the polygonality index, instead of considering only the immediate neighbors of a node to estimate the angle α_i, the *virtual edges* [8] are also taken into account. In case a node i connects to a node j and the latter is linked to a node k (with $i \neq k$), we say that a virtual edge of length 2 is established between nodes i and k. This definition can be immediately extended to further hierarchies. The angles between subsequent virtual edges are taken into account to compute the polygonality index. Therefore, the expression presented in equation (1) can be reformulated in order to include the hierarchical level h, resulting in

$$\Delta_h(k) = \frac{1}{\sum_{i=1}^{N_h(k)} |\alpha_i - \beta_k| + 1}, \tag{2}$$

where $N_h(k)$ is the number of neighbors of the node k at hierarchy h. Figure 2 illustrates the computation of such a polygonality index for three different hierarchical levels of a node.

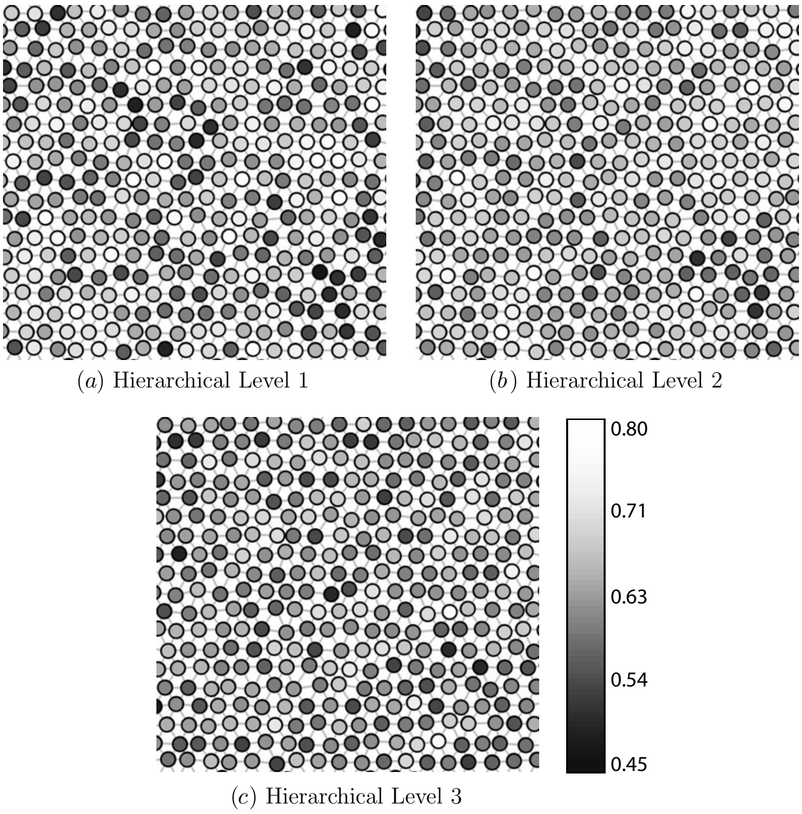


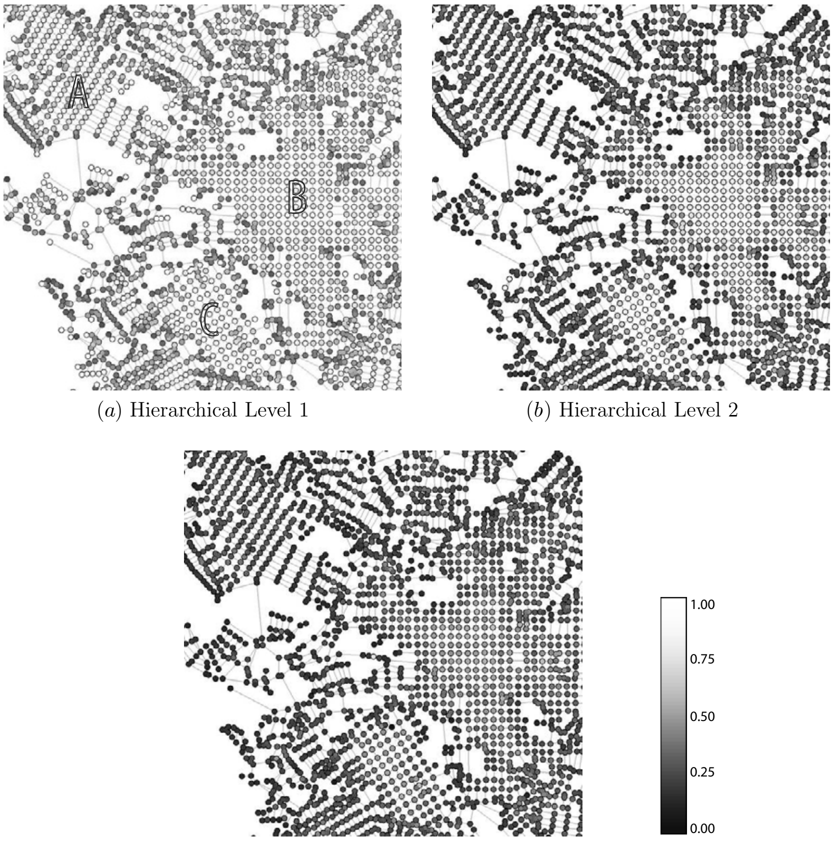
Figure 3. Hierarchical polygonality index for a perturbed hexagonal lattice. The hierarchical level (h), the mean (μ) and standard deviation (σ) values of the polygonality index are (*a*) $h = 1$, $\mu = 0.63, \sigma = 0.07$, (*b*) $h = 2, \mu = 0.64, \sigma = 0.05$ and (*c*) $h = 3, \mu = 0.61, \sigma = 0.05$. The gray scale bar for the polygonality indices is shown at the right-hand side of the figure.

3. Results and discussion

The proposed methodology was employed to analyze two geographical planar networks.

The first network is a synthetic structure whose nodes are arranged in a perfect hexagonal lattice (the connections are determined by considering the nearest neighbors of the nodes). The mean (μ) and standard deviation (σ) values of the polygonality index of the nodes were determined considering different hierarchical levels. The value adopted for β_k varies according to the number of neighbors of the point under analysis ($\beta_k = 2\pi/N_h(k)$). For the first hierarchical level, as the nodes and its neighbors are arranged in a hexagonal way, the polygonality index for all nodes achieves its maximum value ($\mu = 1$ and $\sigma = 0$). In the case of the second hierarchy, the angles defined between successive neighbors of the nodes are constant and equal to $\beta_k = \pi/6$. As a consequence, the polygonality index for each node is also at its maximum value ($\mu = 1$ and $\sigma = 0$). On the other hand, when considering the third hierarchy, the angles between successive neighbors are no longer constant, but the polygonality index is the same for all nodes ($\mu = 0.73$ and $\sigma = 0$).

The potential of the proposed methodology can be clearly verified when the positions of the nodes of the hexagonal lattice are perturbed in order to reduce the overall order, as illustrated by the network shown in figure 3. This perturbation is accomplished by performing



(a) Hierarchical Level 1 (b) Hierarchical Level 2

(c) Hierarchical level 3 Polygonality Index

Figure 4. Polygonality index of the network of the urban streets of São Carlos, Brazil. The hierarchical levels 1 (a), 2 (b) and 3 (c) were considered. The gray levels of the nodes vary accordingly to their polygonality indices as shown in the scale bar. In (a), A, B and C indicate the location of the most regular regions of the network when the first hierarchy is considered.

a small displacement, in a random direction, of the node position in the hexagonal lattice. The polygonality index for this network was determined for the hierarchical levels 1 (figure 3(a), $\mu = 0.63$ and $\sigma = 0.07$), 2 (figure 3(b), $\mu = 0.64$ and $\sigma = 0.05$) and 3 (figure 3(c), $\mu = 0.61$ and $\sigma = 0.05$). In all these images, the gray levels of the nodes indicate the polygonality index accordingly to the scalar bar presented at the right-hand side of figure 3. It is interesting to verify that several nodes have markedly different polygonality indices at different hierarchies, confirming the fact that a node can be perfectly polygonal at the first hierarchy and highly disordered at the second and/or third levels. It is such a complementary characterization of the spatial order around each node which makes the hierarchical polygonality measurement particularly valuable for characterizing spatial order.

The second example considers a network of urban streets (e.g. [9–12]). The network was derived from a map of the streets of the town of São Carlos, Brazil. Image processing techniques were used in order to derive the network. The nodes of the network are located at the crossings of the streets and the connections are given by the streets, resulting in a network with 4537 nodes and 7527 edges. Figure 4(a) shows a representation of a portion of this

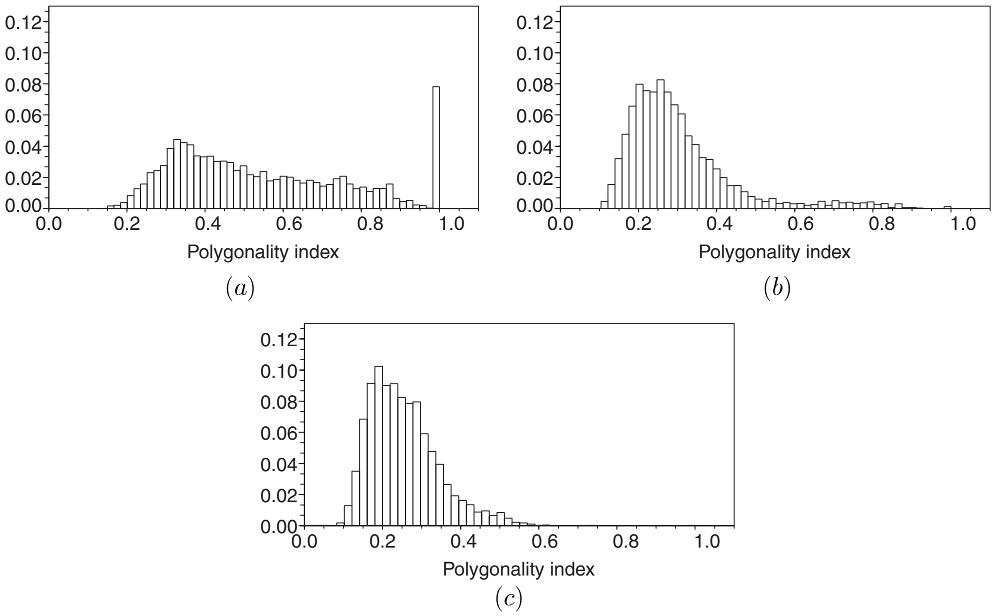


Figure 5. Relative frequency histogram of the polygonality indices of the urban streets of São Carlos. These graphics corresponds to the first (*a*), the second (*b*) and the third (*c*) hierarchical levels.

network. The polygonality index of this network considering $\beta_k = 2\pi/N_h(k)$ was determined for the hierarchical levels 1 (figure 4(*a*)), 2 (figure 4(*b*)) and 3 (figure 4(*c*)). As can be seen in figure 4(*a*) and in the peak in the distribution of the polygonality index in the histogram of figure 5(*a*), three main regions of the town present high spatial organization at the first level (regions A, B and C in figure 4(*a*)). However, most of the nodes have a small polygonality index when the hierarchical levels 2 and 3 are considered, as shown in figures 4(*b*) and (*c*) and the histograms in figures 5(*b*) and (*c*). In both cases, only two regions of the town (regions B and C in figure 4(*a*)) present a high spatial order also at these two higher levels. These regions, which correspond to the center of the more organized groups of nodes, can be considered as the spatially most regular portions of the considered network, as they exhibit high polygonality indices over all the three considered hierarchies.

In order to demonstrate further the discriminative power of the hierarchical polygonality indices for higher levels (i.e. 2 and 3), consider the scatterplot shown in figure 6(*a*). In this scatterplot, the polygonality index of the first hierarchy is shown against the polygonality index of the second hierarchy. Note that a cluster of points is formed at the upper-right quadrant of the scatterplot (dashed rectangle) which could by no means be identified by taking into account only the first polygonality index (i.e. the horizontal axis). The network nodes belonging to this cluster, characterized by a particularly high polygonality index for the first and second hierarchical levels, are shown as white circles in figure 6(*b*), while the other nodes are shown in black. Two of the most organized regions of the town are again clearly identifiable. Observe that one of the regions previously identified (i.e. region A in figure 4(*a*)) does not contain any of the nodes in the cluster in figure 6(*a*). Therefore, this region is spatially ordered only at the first hierarchical level. Such a discrimination between the three regions of high spatial order at the first level would have been completely impossible had not the higher levels been taken into account.

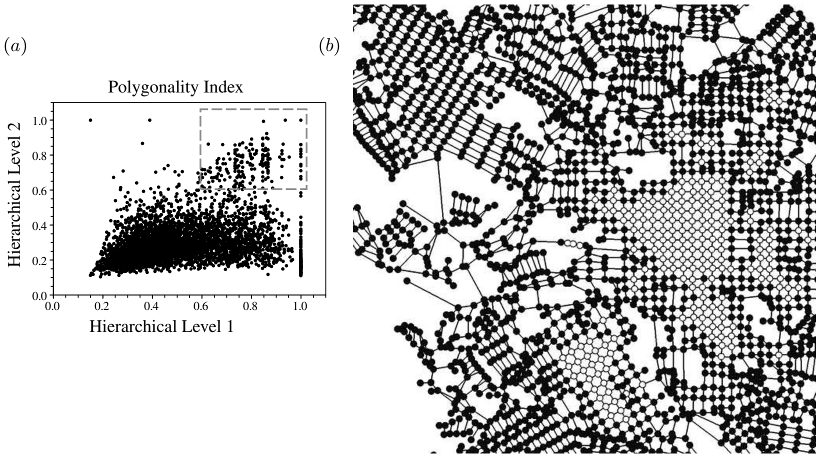


Figure 6. Identification of the most organized regions of the town revealed by the combination of the first two polygonality indices. (*a*) Scatterplot defined by the polygonality indices of the first and second hierarchical levels. The dashed rectangle identifies a cluster of points characterized by high polygonality at both first and second levels. (*b*) Network of urban streets of São Carlos. The nodes of the network which corresponds to the points contained in the cluster in (*a*) are shown in white, while all the other nodes are shown in black.

Finally, in order to compare the results obtained here to other properties of the same network, we also characterized the network of urban streets of São Carlos by using the set of measures proposed by Cardillo *et al* [12]. In their work is proposed the use of the *meshedness coefficient M* [13], a measure able to quantify the connectivity of planar network, which ranges from zero (tree network) to one (fully connected planar graph). In the case of São Carlos, the value obtained was $M = 0.330$, indicating that the town has a relative complex form [12]. The other measures considered in the same work were the number of short cycles (C_l) of length $l = 3, 4, 5$ [14]. These values were normalized with respect to the number of cycles $\left(C_l^{GT}\right)$ of the network formed by the greed triangulation of the nodes [15]. The values obtained for the urban street analyzed in this work were $C_3/C_3^{GT} = 0.031$, $C_4/C_4^{GT} = 0.088$ and $C_5/C_5^{GT} = 0.017$. These values can be compared with other cities presented in [12]. Observe that the major number of cycles refer to C_4/C_4^{GT}, indicating a dominance of squares with respect to triangles in this network. In addition, when comparing this result with the polygonality index over all hierarchies (figure 4), it can be noted that the most regular regions of the town are comprised of squares.

4. Concluding remarks

Complex networks have been extensively used in order to represent the connectivity structure of a wide range of complex systems, from protein interaction to the Internet (e.g. [16, 17]). While a growing number of measurements have been proposed and applied in order to characterize such networks, substantially fewer works have addressed the equally important problem of quantifying the spatial properties of geographical networks. Being characterized by the fact that their nodes possess well-defined positions, geographical networks can be used to represent a large variety of real complex systems, including power distribution, transportation systems (e.g. airports and railways), as well as urban streets.

The concept of polygonality has been used [6, 18] to provide an overall quantification of the spatial organization of systems of objects. By establishing a neighborhood system among the constituent points (achieved through Voronoi tessellation), it becomes possible to define the neighbors of each point. The polygonality index quantifies the uniformity of the successive angles around each point. The current work extended the concept of polygonality to reflect the neighborhood defined by the network connectivity as well as to reflect a wider neighborhood around each node in a geographical planar network. By doing so, it is possible to derive a better understanding of the spatial order along successive spatial scales, therefore reflecting the context around node in a more comprehensive way. This has been achieved by making use of the concept of hierarchical neighborhoods, which had been previously explored for the characterization of topological aspects of complex networks (e.g. [19]). In addition to presenting the extended polygonality index, we also illustrated the potential of this new measurement with respect to a synthetic situation (perfect and perturbed hexagonal lattice) as well as a real network defined by the streets and street intersections of a Brazilian town (São Carlos). The results obtained for the latter application make it clear that the two most regular regions of the network (i.e. those surrounded by several spatially regular neighborhoods—regions B and C) presented higher index values for a wider range of hierarchies, which would be otherwise overlooked by the traditional polygonality index. Another region identified in the town exhibits high values of the first level polygonality but low values at the second and third levels (region A). These results were also complemented and compared to the meshedness coefficient [13] and the number of short cycles [14].

All in all, the suggested hierarchical polygonality indices allowed a comprehensive characterization of the spatial organization of geographical complex networks. They can also be compared with several other measurements related to the dynamics and connectivity of the network, paving the way to obtaining new insights about relationships between the topological and geometrical properties of these networks.

Acknowledgments

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